

Constructive Methods for Localization and Eigenvalue Statistics

John Imbrie

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Overview

Convergent expansions for eigenvalues and eigenvectors lead to new insights in many-body and single-body quantum systems with disorder. I will review recent work elucidating the way randomness localizes eigenfunctions, smooths out eigenvalue distributions, and produces eigenvalue separation.

References:

MBL: [arXiv:1403.7837](#) (JSP 2016), [arXiv:1605.03003](#) (PRL 2016)

Review on LIOMs (with Ros, Scardicchio): [arXiv:1609.08076](#)

Anderson model version: [arXiv:1406.2957](#) (CMP 2016)

Level Spacing (with Mavi): [arXiv:1506.06692](#) (JSP 2016)

Outline

A result on many-body localization (MBL) under an assumption on level statistics.

Local Integrals of Motion (LIOMs) and eigenstate labels.

New approaches to level statistics (single-body Hamiltonians):

“Internal” disorder

“Environmental” disorder

A Model for Many-Body Localization

Random field, random transverse field, random exchange Ising model on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^K h_i S_i^z + \sum_{i=-K}^K \gamma_i S_i^x + \sum_{i=-K-1}^K J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable.

Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

This Hamiltonian is close to one that is diagonal in the basis given by tensor products of S_i^z eigenstates. Indeed, the MBL transition in this model can be thought of as a many-body version of what happens for a single spin. Taking Λ to be a single site, the Hamiltonian reduces to

$$\begin{pmatrix} h & \gamma \\ \gamma & -h \end{pmatrix}.$$

For small γ , the eigenvectors are close to $(1, 0)$ and $(0, 1)$, whereas for large γ , they are close to the fully hybridized vectors $(1, 1)$ and $(1, -1)$.

The following result demonstrates (under an assumption on level statistics) the existence of an MBL phase where the eigenstates of H resemble the basis vectors, *i.e.* they are largely unhybridized.

Theorem

Assume that for some ν, C , the eigenvalues of H in boxes of size n satisfy

$$P\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n, \quad (1)$$

for all $\delta > 0$ and all n . Then there exists a $\kappa > 0$ such that for γ sufficiently small,

$$\mathbb{E} \text{Av}_\alpha |\langle S_0^Z \rangle_\alpha| = 1 - O(\gamma^\kappa),$$

for all Λ . Here \mathbb{E} denotes the disorder average, $\langle \cdot \rangle_\alpha$ is the expectation in the eigenstate α , and Av_α is an average over α .

Remark. The level-statistics assumption (1) says that with high probability the minimum level spacing should be no smaller than an exponential in the volume. Physically, one expects to see (1) satisfied with $\nu = 1$ in a localized phase (Poisson statistics) or with $\nu > 1$ in a thermalized phase (repulsive statistics).

In the course of the proof, we obtain:

- (i) Existence of a labeling system for eigenstates by spin/metaspin configurations, with metaspins needed only on a dilute collection of resonant blocks. (Spin variables used to label basis vectors can also be used to label the exact eigenstates, but the correspondence becomes somewhat arbitrary in resonant regions, so we use the term “metaspin” instead.)
- (ii) Faster-than-power-law decay of the probability of resonant blocks, which implies their diluteness. (This is critical to the whole concept of a labeling system – without it the labeling system would lose its meaning.)
- (iii) Diagonalization of H via a sequence of local rotations defined via convergent graphical expansions with exponential bounds. (Locality means that graphs depend only on the random variables in their immediate vicinity.)

- (iv) Bounds establishing closeness of expectations of local observables in any eigenstate to their naïve ($\gamma = 0$) values, when observables are not in resonant regions. (This makes precise the idea that eigenstates resemble the basis vectors.)
- (v) Corollary: violation of Eigenstate Thermalization Hypothesis
- (vi) Exponential decay of connected correlations in eigenstates, except on a set of rapidly decaying probability. (This shows the exponential loss of entanglement with distance for the subsystems associated with the observables.)

Local Integrals of Motion (LIOMs)*

Diagonalization of H with quasilocal rotations means that the interacting eigenstates are “deformed” versions of the basis vectors (tensor products of $(0,1)$ or $(1,0)$ at each site). This means that the eigenstates are “labeled” by the corresponding basis vector. This allows a construction of LIOMs:

Let Ω be the rotation that diagonalizes H . Then $\tilde{H} = \Omega^* H \Omega$ is diagonal. The columns of Ω are the eigenvectors. As they are labeled by spin configurations σ , one may use σ as the column index for \tilde{H} . Thus \tilde{H} is a diagonal matrix that can act on vectors in the original tensor product basis.

Note that the spin operators S_i^Z are also diagonal in this basis. Therefore, $[\tilde{H}, S_i^Z] = 0$ for each $i \in \Lambda$. Transferring the rotations to the spin operators, one obtains that $[H, \tau_i] = 0$, where

$$\tau_i \equiv \Omega S_i^Z \Omega^*.$$

* See also Serbyn, Papić, Abanin, PRL 2013; Huse, Oganesyan PRB 2014.

Analogous construction: Labeling for the Anderson model

Let $H_A = -\gamma\Delta + v$, where Δ is the lattice Laplacian, $0 < \gamma \ll 1$, and v is multiplication by a random potential v_x , $x \in \Lambda \subset \mathbb{Z}^d$.

In arXiv:1406.2957, I diagonalized H_A with a sequence of quasilocal rotations, obtaining eigenstate labelling and exponential decay of the EFC. This was a test case for the many-body construction.

As above, one finds that $\tilde{H}_A = \Omega^* H_A \Omega$ is diagonal, and $\Omega_{yx} = \psi_x(y)$. Let $|x\rangle$ denote the basis vector which is 1 at x and 0 elsewhere. Then $|x\rangle\langle x|$ is the projection onto functions supported at x . As it is a diagonal operator, it commutes with \tilde{H}_A . Define

$$\tilde{\tau}_x \equiv \Omega|x\rangle\langle x|\Omega^* = |\psi_x\rangle\langle\psi_x|.$$

Then $[H_A, \tilde{\tau}_x] = 0$. In this case the LIOM is simply the projection onto the subspace spanned by the eigenstate with label x .

Related work: Elgart, Klein, arXiv:1509.08521 and their talks yesterday; Klein, Tseng, arXiv:1605.03637.

How to prove a Minimum Level Spacing condition?

Minimum level spacing is physically reasonable, but hard to prove mathematically. It is needed to control interactions between resonant regions:



New methods for establishing minimum level spacing urgently needed!

First, let us develop the necessary tools in the single-body context. One method involves exploiting the randomness internal to resonant regions (“internal disorder”). The other uses randomness in the vicinity of resonant regions (“environmental disorder”).

Disorder internal to resonant regions leads to level separation

In 1506.06692, Mavi and I prove a minimum level spacing condition for a random block Anderson model.

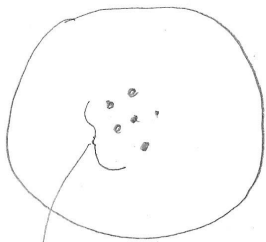
Each site consists of a 2×2 block $\begin{pmatrix} h_i & 1 \\ 1 & -h_i \end{pmatrix}$

We use a sequence of Schur complements to restrict the Hilbert space to modes with energies in small windows about some E .

The effective Hamiltonian for n modes is linear in the potentials $\{h_x\}$. Its discriminant $\prod_{\alpha < \beta} (\lambda_\alpha - \lambda_\beta)^2$ is a polynomial of degree $2n^2$ in $\{h_x\}$, which limits the measure of sets where it is near zero.

For a many-body Hamiltonian, the degree would be exponential in n , which weakens the probability bounds, so this method may not be sufficient.

Internal disorder \Rightarrow { density of states
level spacing



Resonant Block: n sites

Discriminant
= polynomial in $\{h_i\}$
degree n^2



$$P(\min_{\alpha < \beta} |E_\alpha - E_\beta| < \delta) \leq \delta^{1/n^2}$$

$$\max_n \varepsilon^n \delta^{1/n^2} = e^{|\log \varepsilon|^{2/3} |\log \delta|^{1/3}}$$

Result

Theorem. *There exists a constant $b > 0$ such that for γ sufficiently small,*

$$P\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq |\Lambda| \exp\left(-b |\log \gamma|^{3/4} |\log \delta|^{1/4}\right),$$

for all $\delta > 0$.

Related work:

Elgart, Shamis, Sodin, JEMS 2014: localization

Elgart, Schmidt, arXiv:1306.3459: Minami-type estimates for models e.g. with blocks $\begin{pmatrix} u & v \\ v & -u \end{pmatrix}$

Environmental Disorder: Discrete disorder distributions

Consider again the Anderson model in \mathbb{Z}^d , $d \geq 2$:

$$H = -\gamma\Delta + v.$$

Let each v_x have a uniform distribution on $\{0, \frac{1}{N-1}, \frac{2}{N-1}, \dots, 1\}$, with $N \gg 1$.

For $N = 2$, this is the Anderson-Bernoulli model.

Related work: Bourgain, Kenig, Invent. Math. 2005: Localization for the continuum version of Anderson-Bernoulli ($N = 2$, $\mathbb{Z}^d \rightarrow \mathbb{R}^d$) for energies near the bottom of the spectrum, $E = 0$.

Large N Result (work in progress)

Let $I_\delta(E)$ denote the interval $[E - \delta, E + \delta]$, and let $\mathcal{N}(I)$ denote the number of eigenvalues of H in I .

Result (mostly complete). *Choose a sufficiently large p . Then for N sufficiently large (depending on p) and γ sufficiently small (depending on N),*

$$\mathbb{E} \mathcal{N}(I_\delta(E)) \leq |\Lambda| (\tfrac{1}{7} \log_\gamma \delta)^{-p},$$

and

$$P\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq |\Lambda|^2 (\tfrac{1}{7} \log_\gamma \delta)^{-p}.$$

for any rectangle Λ and any $\delta \in [\gamma^{\text{diam}(\Lambda)/2}, 1]$.

Also, $\sum_\alpha |\psi_\alpha(x) \psi_\alpha(y)|$ decays exponentially, and its disorder average is bounded by $C|x - y|^{-p}$.

Disorder in environment \Rightarrow $\left\{ \begin{array}{l} \text{density of states} \\ \text{level spacing} \end{array} \right.$

